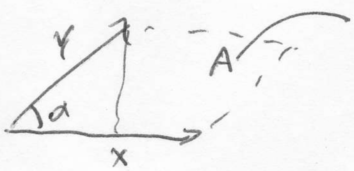
Sectional curvature

(M,g) $\qquad g$ - Riemannian signature.

$p \in M$ and let $\sigma \subset T_pM$ ← 2-dimensional vector subspace.

$X, Y \in T_pM$ s.t. $\mathrm{Span}(X,Y) = \sigma$.



$$A(X,Y) = |X||Y|\sin\alpha = |X \wedge Y|$$

$$|X\wedge Y|^2 = |X|^2|Y|^2(1-\cos^2\alpha) =$$
$$= |X|^2|Y|^2 - |X|^2|Y|^2\cos\alpha =$$
$$= g(X,X)g(Y,Y) - g(X,Y)^2$$

$$|X\wedge Y|^2 = g(X,X)g(Y,Y) - g(X,Y)^2 > 0 \text{ if } X, Y \text{ linearly indep.}$$

Def

Given $\sigma \subset T_pM$ and $X, Y \in T_pM$ s.t. $\sigma = \mathrm{span}(X,Y)$

a sectional curvature of σ at p is a reall number

$$K(X,Y) = \frac{g(R(Y,X)X,Y)}{|X\wedge Y|^2} = -\frac{g(R(X,Y)X,Y)}{|X\wedge Y|^2}$$

Fact K depends only on σ and not on the choice of the basis X, Y in it.

$$K(X,Y) = K(\sigma) \quad \text{where} \quad \sigma = \mathrm{Span}(X,Y).$$

Proof

the following transformations:

$$(X,Y) \longmapsto (Y,X)$$
$$(X,Y) \longmapsto (\lambda X, Y)$$
$$(X,Y) \longmapsto (X+\lambda Y, Y)$$

generate the most general linear transformation in 2-dimensions. (by iterations)

But $K(X,Y)$ is unchanged by any of these $\square$.

Thm

Knowledge of $K(X,Y)$ for all $X,Y \in TM$ determines the curvature $R(X,Y)$.

Proof

Given $K(X,Y)$ we know that it is determined by R - which is the curvature of g.

Suppose that there exist R', a tensor

$$R' : V \times V \times V \longrightarrow V \quad \text{s.t.} \quad R' \neq R$$

$$R'(X,Y)Z + R'(Z,X)Y + R'(Y,Z)X = 0$$
$$R'(X,Y) = -R'(Y,X)$$
$$g(R'(X,Y)Z,T) = -g(R'(X,Y)T,Z)$$
$$g(R'(X,Y)Z,T) = g(R'(Z,T)X,Y)$$

and for which $K(X,Y) = \dfrac{g(R(Y,X)X,Y)}{|X \wedge Y|^2} = \dfrac{g(R'(Y,X)X,Y)}{|X \wedge Y|^2}$

for all $X,Y \in TM$. This means that

$$g(R(X,Y)X,Y) = g(R'(X,Y)X,Y) \quad \forall X,Y \in TM$$

We denote by $(X,Y,Z,T) = g(R(X,Y)Z,T)$

and $(X,Y,Z,T)' = g(R'(X,Y)Z,T)$.

and we have

$$(X,Y,X,Y) = (X,Y,X,Y)' \text{ by our assumption.}$$

$$\Rightarrow (X+Z,Y,X+Z,Y) =$$

$$= (X,Y,X,Y) + 2(X,Y,Z,Y) + (Z,Y,Z,Y)$$

$$= (X,Y,X,Y)' + 2(X,Y,Z,Y)' + (Z,Y,Z,Y)'$$

hence $(X,Y,Z,Y) = (X,Y,Z,Y)'$

$$\Rightarrow (X,Y+T,Z,Y+T) = (X,Y,Z,Y) + (X,Y,Z,T) + (X,T,Z,Y) + (X,T,Z,T)$$

$$= (X,Y,Z,Y)' + (X,Y,Z,T)' + (X,T,Z,Y)' + (X,T,Z,T)'$$

$$\Rightarrow (X,Y,Z,T) + (X,T,Z,Y) = (X,Y,Z,T)' + (X,T,Z,Y)'$$

$$A = \boxed{(X,Y,Z,T) - (X,Y,Z,T)' = (Y,Z,X,T) - (Y,Z,X,T)'}$$

cyclic permutation

$$\sigma A = (Z,X,Y,T) - (Z,X,Y,T)' = (X,Y,Z,T) - (X,Y,Z,T)' = A$$

$$\sigma^2 A = (Y,Z,X,T) - (Y,Z,X,T)' = (Z,X,Y,T) - (Z,X,Y,T)' = \sigma A = A$$

$$0 = A + \sigma A + \sigma^2 A = 3A \Rightarrow A = 0 \Rightarrow \boxed{(X,Y,Z,T) = (X,Y,Z,T)'}$$

Which means that $R = R'$ □.

What are spaces of constant $\overbrace{\text{curvature}}^{\text{SECTIONAL}}$?

$$K_0 = \frac{-(X,Y,X,Y)}{|X\wedge Y|^2} \Rightarrow (X,Y,X,Y) = -K_0 |X\wedge Y|^2$$

Define R' by:

$$g(R'(X,Y)Z,T) = K_0\left[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)\right]$$

$$\| $$

$(X,Y,Z,T)'$. It satisfies all the four symmetries of Riemann.

Then:

$$(X,Y,X,Y)' = K_0\left(g(Y,X)g(X,Y) - g(X,X)g(Y,Y)\right) =$$

$$= -K_0 |X\wedge Y|^2$$

$$\Rightarrow \quad (X,Y,X,Y)' = (X,Y,X,Y) \qquad \forall X,Y \in TM$$

$$\Rightarrow \quad R = R'$$

$$R_{\mu\nu\rho\sigma} X^\rho Y^\sigma Z^\nu T^\mu = K_0\left[g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}\right]$$

$$\Rightarrow \boxed{R_{\mu\nu\rho\sigma} = K_0\left(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}\right)}$$

$$\Rightarrow \begin{pmatrix}\text{spaces of constant} \\ \text{sectional} \\ \text{curvature}\end{pmatrix} \Leftrightarrow \begin{pmatrix}\text{Weyl} \equiv 0 \\ \text{Traceless Ricci} \equiv 0\end{pmatrix}$$

$\square$.